COS 140: Foundations of Computer Science

Hamming Codes

Fall 2018

Homework, etc.

- New book chapter (25) online
- Slides online at website
- Exercises at end of the chapter, due 12/5.

The problem

- Want a way to determine if data is correct:
  - from memory
  - across the network
  - from an I/O device
- Even better, want a way to fix the data if it is incorrect

Why Not Parity Bits?

- Parity bit just says “an error occurred” – not *where* it occurred.
- For messages:
  - ○ If only know an error occurred, will need to resend message.
  - ○ It can take a great deal of time to resend, so better if you can fix the error.
- For memory: can’t do equivalent of resending message!
- Know from RAID that can correct the error if know the location of the bit.

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- Problem: More than doubles size of memory!
- (Question: what is this scheme equivalent to when using even parity?)

Hamming Codes

- Example of an *error-correcting code*: way to correct errors in memory, in transmission.
- Used in computer networks, internal memory and elsewhere in the computer.
- Goal: Identify which bit has the error, but use fewer bits to do it than the parity scheme

Hamming Codes: Basic idea

- Associate parity bits with different *subsets* of the data
- Any single bit is uniquely linked to some group of parity bits.
- Each of those bits will indicate an error when that bit is incorrect

Associating Data with Parity Bits

1 1 1 0

Data to be checked/corrected

Overview

General idea

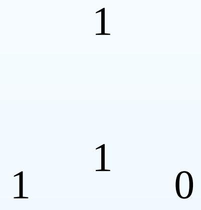
● Associating Data with Parity Bits

● Correcting a Bit

Hamming Codes

Error Correction

Associating Data with Parity Bits



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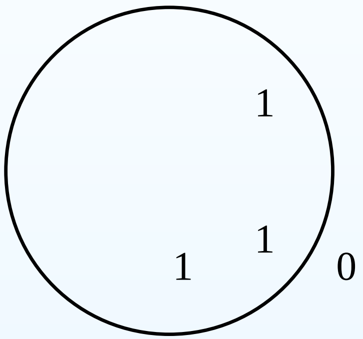

Divide data into subsets

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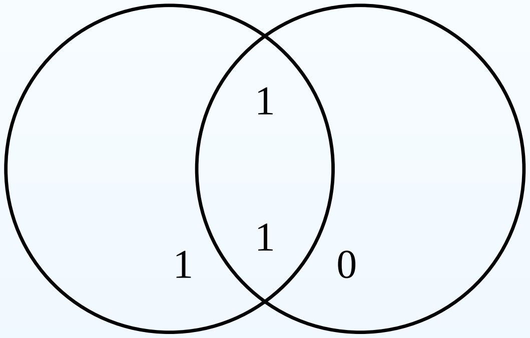
Associating Data with Parity Bits



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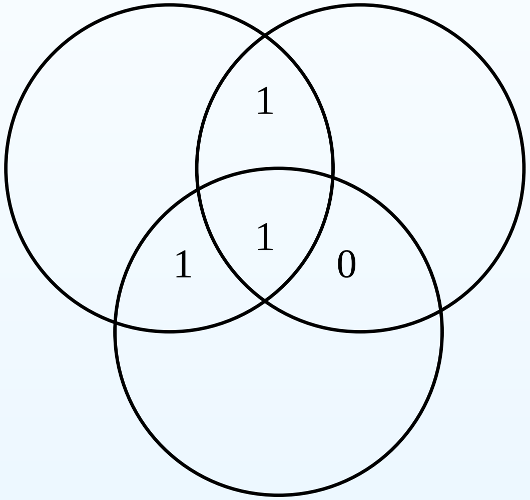

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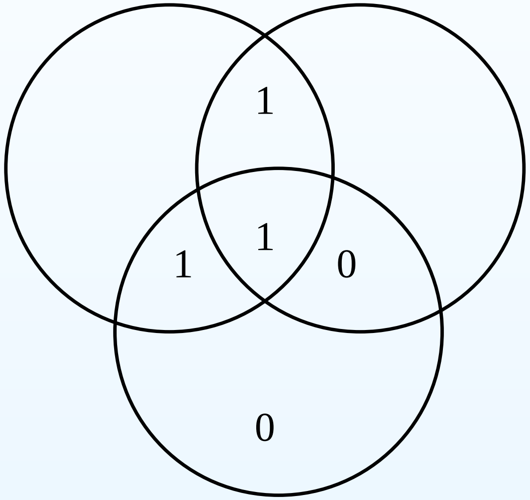


Compute even parity for bits in each circle

Overview

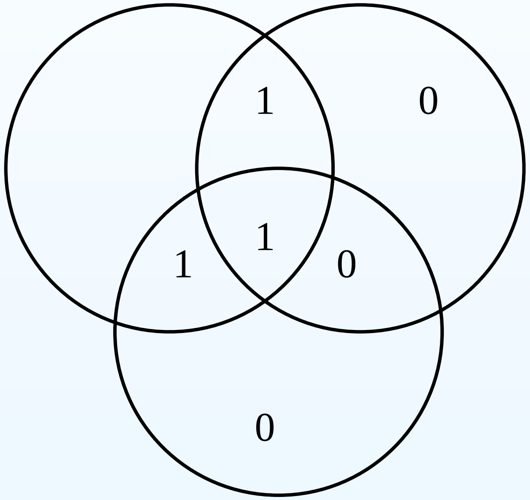
General idea

- Associating Data with Parity Bits
- Correcting a Bit

Hamming Codes

Error Correction

Associating Data with Parity Bits



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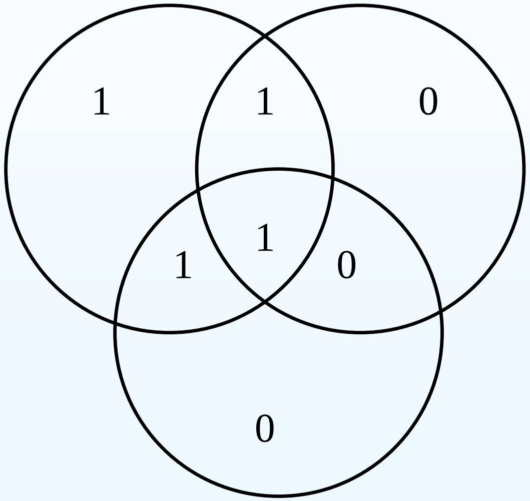

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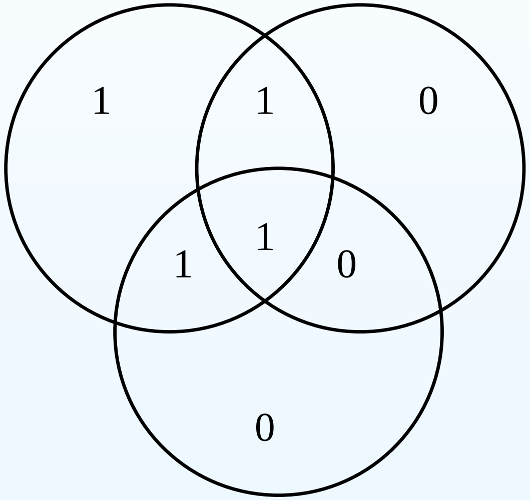
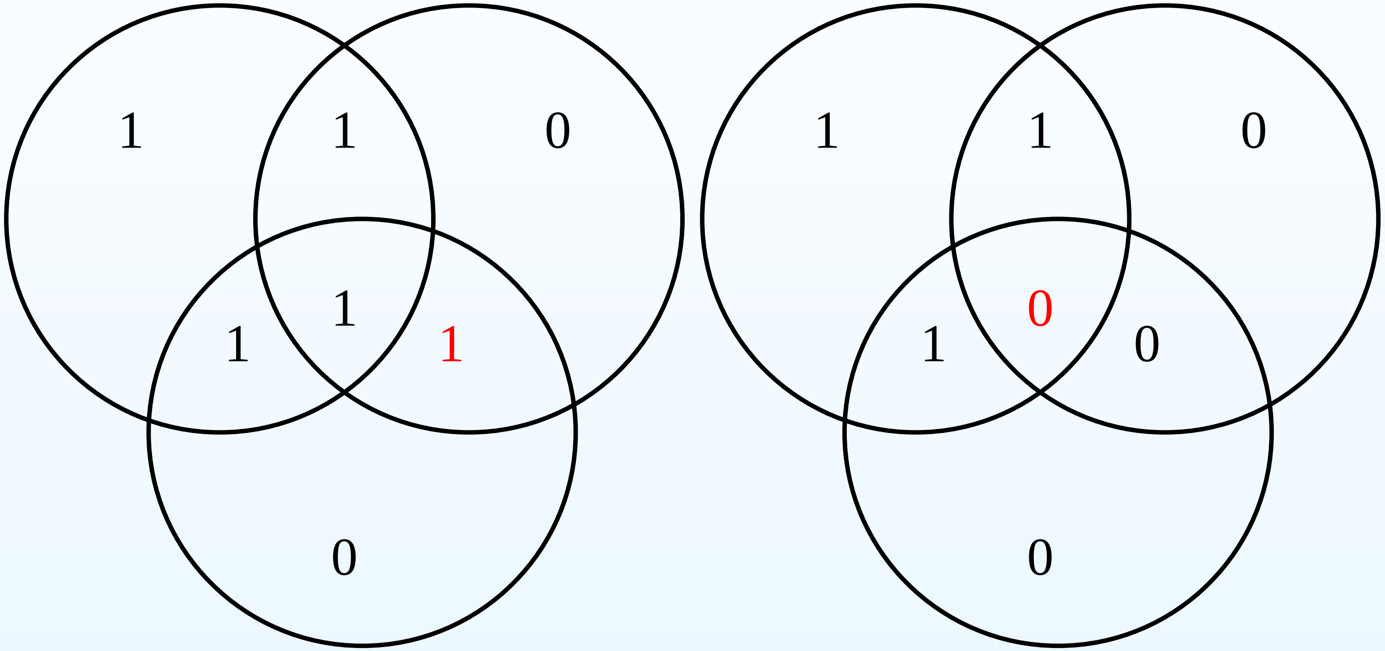


Now each bit is checked by unique combination of parity bits

Correcting a Bit



- When get data, check all the parity bits.
- Left figure: two parity bits wrong.
- Right: all three wrong.
- Data to be corrected is at the intersection of all sets where parity bit is wrong.

Hamming Codes in Transmitted Data

- Have data in a stream, not in Venn diagrams.
- Divide data into segments that can be checked with Hamming codes.
- Need on the order of $\log_2 n$ checkbits for n bits.
- Add one check bit each time the number of data bits doubles.
- So: segments should be fairly large (but small enough that don't have unused bits).

Overview

General idea

Hamming Codes
- Transmitted data
- Data subsets
- Parity
- Code for 8-bit data

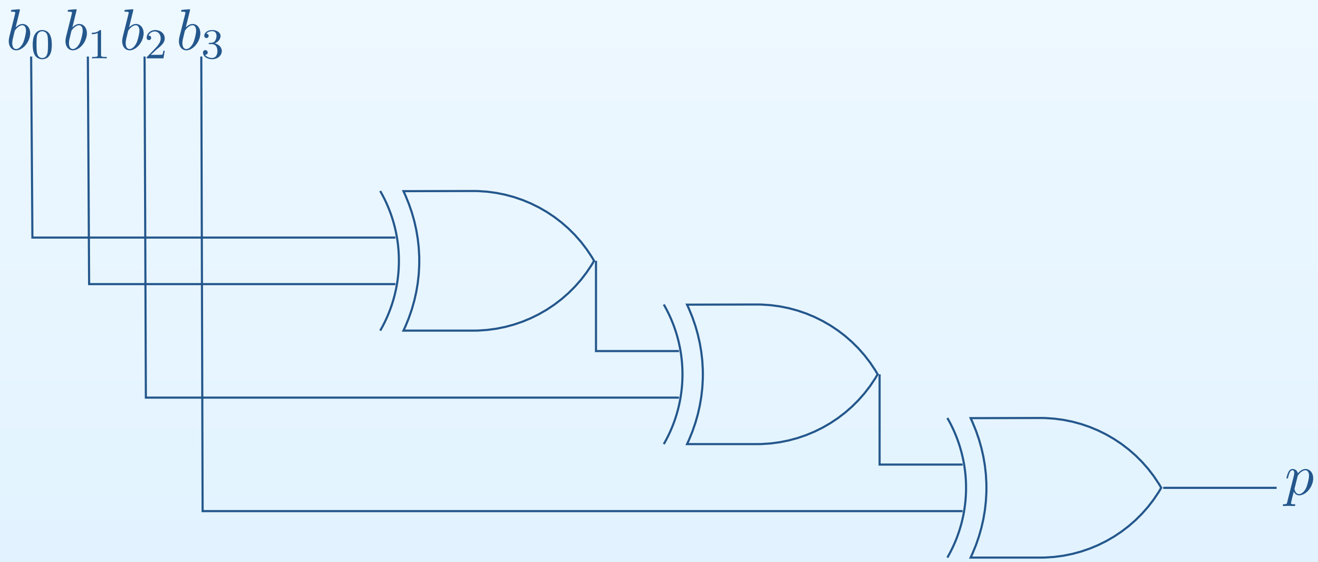
Error Correction

Dividing the Data into Subsets

- Recognize that each bit has a unique position in the stream (starting at position 1).
- The position can be written as a binary number.
- In the binary representation of the position, some digits are 1 and some are 0.
- Put check bits in the positions that correspond to powers of 2 (1,2,4,8, etc.)
- Put data bits in the other positions.
- Check bits check parity for all positions that have a 1 for the corresponding digit (e.g., the bit at position 12 (1100) is checked by parity bits at positions 8 and 4).
- Because all binary numbers are combinations of different powers of 2, each position is checked by a different set of bits.
- Need enough check bits to “address” data bits plus check bits.

Aside: Computing parity

- Computing parity is trivial!
- Suppose parity bit p checks bits b_0–b_3
- Using single-bit addition, $p = b_0 + b_1 + b_2 + b_3$
- Example: Data = 1011: 1 + 0 + 1 + 1 = 11, but no carry, so $\Rightarrow$ 1 – correct parity bit
- In hardware:
  - ○ Could use half-adder – but don't need the carry
  - ○ So just $p = b_0 \oplus b_1 \oplus b_2 \oplus b_3$



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Hamming Code for 8 Bits of Data

- Check (parity) bit locations: 2^0, 2^1, 2^2, 2^3, ⇒ word length = 12 bits
- Example: 1101 1110

0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100
C1	C2	D	C4	D	D	D	C8	D	D	D	D
		1		1	0	1		1	1	1	0

- Data bit relations to check bits:

Data bit	Location in Word	Check Bits
1	3 (0011)	C1, C2
2	5 (0101)	C1, C4
3	6 (0110)	C2, C4
4	7 (0111)	C1, C2, C4
5	9 (1001)	C1, C8
6	10 (1010)	C2, C8
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Example of Hamming Code Correction

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General idea

Hamming Codes

Error Correction

● Example

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Error in bit 7: 1101 1100

0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100
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Computed parities:

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Parity errors will appear when computing
C1, C2, C8 ⇒ bit 1011 in error.