COS 140: Foundations of Computer Science

Boolean Algebra

Fall 2018

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Homework and announcements!

- ☐ Reading: Chapter 5
- ☐ Homework: exercises 1–6
  - – Exercise 7 for extra credit
  - – Due: 1 week + 1 class from today (i.e., 9/21)
- ☐ Don't forget – **recitation this week**!
- ☐ Slides: online now

Introduction

Problem

- ☐ Computers compute digital logic functions
- ☐ Need some way to describe those functions
- ☐ For some ordinary mathematical functions, algebra works well
- ☐ But the functions we want aren't numeric, but give true/false (1/0) values
- ☐ What can we use?

Boolean algebra

- Boolean algebra is analogous to "regular" algebra, but for true/false values
- Well-known, well-developed mathematical foundation for digital logic.
- Provides a formalism for specifying the functions that we wish to have performed.
- Provides a mechanism for proving circuits are equivalent.
- Named for George Boole, a 19^{th}-century mathematician.

Operators

Boolean operators correspond to gates and have same truth tables as corresponding gate.

- NOT: NOT A, $\neg A$, $\overline{A}$, A'
- AND: A AND B, $A \cdot B$, AB, $A \wedge B$
- OR: A OR B, $A + B$, $A \vee B$
- NAND: A NAND B, $A|B$, $A \uparrow B$, $\overline{AB}$
- NOR: A NOR B, $A \downarrow B$, $\overline{(A+B)}$
- XOR: A XOR B, $A \oplus B$

Boolean Algebra Expressions

- What is an *expression*?
- Value of an expression $\Leftarrow$ depends on the values of the variables
- Evaluate expression: assign values to variables, performing the operations
- Can create an expression for a function by determining when the function should be 1, then writing an expression that is 1 in only those cases.

Example: Create a 3-variable expression that equals 1 when all of the inputs are 1, or when one, and only one, input is 0.

Operator Precedence for Boolean Algebra

- Subexpressions inside of parentheses, beginning with innermost parentheses.
- NOT
- AND
- OR
- Evaluate A NAND B as NOT($A \wedge B$) and A NOR B as NOT($A \vee B$)
- Evaluate subexpressions of equal precedence from left to right.

Proofs

Proofs of Equivalence

- Two ways: truth tables and algebraic substitution
- Truth table method: if truth table for expression A same as for expression B, then $A \equiv B$
- Algebraic substitution method:
  - Use laws of Boolean algebra to transform one expression into the other
  - Proofs have to be convincing to others
  - Have to provide enough detail to show how one step follows from another
  - Have to provide justification for each step

Proof by Truth Table

Best to have column for each input and results of each operator

Prove that $A \oplus B$ is equivalent to $(\overline{A}B) + (A\overline{B})$

A	B	$A \oplus B$	$\overline{A}$	$\overline{B}$	$\overline{A}B$	$A\overline{B}$	$(\overline{A}B) + (A\overline{B})$
0	0	0	1	1	0	0	0
0	1	1	1	0	1	0	1
1	0	1	0	1	0	1	1
1	1	0	0	0	0	0	0

Proof by Algebraic Substitution

- If you are trying to prove that expression 1 is equivalent to expression 2:
  - Start with one of the expressions, let's say 1.
  - Change it into another expression (say $1'$) using an *identity postulate* (law).
  - Continue the process with $1'$ until you arrive at 2.
- You must justify *every* change to the current expression by listing the identity postulate used.
- Some identity postulates: double negation law, identity law, null law, idempotent law, and inverse law

Laws

Double Negation Law

- $A = \overline{\overline{A}}$

Identity Law

- AND form: $1A = A$
- OR form: $0 + A = A$

Null Law

- AND form: $0A = 0$
- OR form: $1 + A = 1$

Idempotent Law

- AND form: $AA = A$
- OR form: $A + A = A$

Inverse Law

- AND form: $A\overline{A} = 0$
- OR form: $A + \overline{A} = 1$

Example: Finding Equivalent Circuits

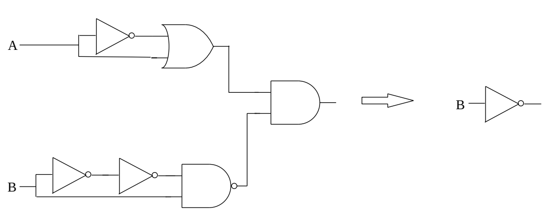
Starting with the following expression, find the equivalent expression that uses the least number of gates (has the smallest number of boolean operators) and the least number of inputs

$$(A\ +\ \overline{A})\overline{(B\overline{\overline{B}})}$$

- ☐ Don't know what the final result will be so use algebraic substitution instead of truth tables
- ☐ Follow your intuition about what should be the case and what makes sense, then justify with a law.
- ☐ In other words, have a plan based on what makes sense.
- ☐ Sometimes you need to try things to make the plan.

Example: Finding Equivalent Circuits

$(A\ +\ \overline{A})\overline{(B\overline{\overline{B}})}$
$(A\ +\ \overline{A})\overline{(BB)}$
$(A\ +\ \overline{A})\overline{B}$
$1\overline{B}$
$\overline{B}$



Commutative Law

- AND form: $AB = BA$
- OR form: $A + B = B + A$

Associative Law

- AND form: $(AB)C = A(BC)$
- OR form: $(A + B) + C = A + (B + C)$

Distributive Law

- AND form: $A(B + C) = AB + AC$
- OR form: $A + BC = (A + B)(A + C)$
- ...or $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$

Absorption Law

- AND form: $A(A + B) = A$
- OR form: $A + AB = A$

DeMorgan's Law

- AND form: $\overline{AB} = \overline{A} + \overline{B}$
- OR form: $\overline{A+B} = \overline{A} \cdot \overline{B}$

Aside: recall that $\overline{AB} = A$ NAND B, and $\overline{A+B} = A$ NOR B
NOTE:

$$\overline{AB} \neq \overline{A}\,\overline{B}$$

$$\overline{A+B} \neq \overline{A} + \overline{B}$$

Using the Laws

How would you prove DeMorgan's Law?

What approach would you use? Why?

Example: Proof by Algebraic Substitution

Prove $\overline{A}B \; + \; A\overline{B} \; = \; \overline{(\overline{\overline{A}B})(\overline{A\overline{B}})}$

- $\overline{(\overline{\overline{A}B})(\overline{A\overline{B}})}$
- $\overline{\overline{(\overline{A}B)}} + \overline{\overline{(A\overline{B})}}$ *DeMorgan's Law*
- $(\overline{A}B) + \overline{\overline{(A\overline{B})}}$ *Double Negation*
- $(\overline{A}B) + (A\overline{B})$ *Double Negation*
- $\overline{A}B + A\overline{B}$ *Def. of parentheses/precedence*

Example: Proof by Algebraic Substitution

Prove $\overline{A}B \; + \; A\overline{B} \; = \; \overline{(\overline{\overline{A}B})(\overline{A\overline{B}})}$

$$
\begin{array}{ll}
\overline{(\overline{\overline{A}B})(\overline{A\overline{B}})} & \\
\overline{XY} & \textit{Let } X = (\overline{\overline{A}B}),\ Y = (\overline{A\overline{B}}) \\
\overline{X} + \overline{Y} & \textit{DeMorgan's Law} \\
\overline{\overline{(\overline{A}B)}} + \overline{\overline{(A\overline{B})}} & \textit{Substitution for } X,\ Y \\
(\overline{A}B) + \overline{\overline{(A\overline{B})}} & \textit{Double Negation} \\
(\overline{A}B) + (A\overline{B}) & \textit{Double Negation} \\
\overline{A}B + A\overline{B} & \textit{Def. of parentheses/precedence}
\end{array}
$$

Example: Proof by Algebraic Substitution

Prove $\overline{A}B \ + \ A\overline{B} \ = \ \overline{(\overline{\overline{A}B})(\overline{A\overline{B}})}$

- □ $\overline{A}B \ + \ A\overline{B}$
- □ $\overline{\overline{\overline{A}B \ + \ A\overline{B}}}$ *Double Negation*
- □ $\overline{(\overline{\overline{A}B})(\overline{A\overline{B}})}$ *DeMorgan's Law*

Example: Another Proof by Algebraic Substitution

Prove $ACB \ + \ C(B \ + \ C) \ = \ C$

- □ $ACB \ + \ C(B \ + \ C)$
- □ $ACB \ + \ CB \ + \ CC$ *Distributive Law*
- □ $ACB \ + \ CB \ + \ C$ *Idempotent Law*
- □ $CB \ + \ ACB \ + \ C$ *Commutative Law*
- □ $CB \ + \ CBA \ + \ C$ *Commutative Law*
- □ $CB \ + \ C \ + \ CBA$ *Commutative Law*
- □ $CB \ + \ C$ *Absorption Law*
- □ $C \ + \ CB$ *Commutative Law*
- □ C *Absorption Law*

Example: Another Proof by Algebraic Substitution

Alternate proof of $ACB + C(B + C) = C$

- □ $ACB + C(B + C)$
- □ $ACB + CB + CC$ *Distributive Law*
- □ $ACB + CB + C$ *Idempotent Law*
- □ $ABC + CB + 1C$ *Identity Law*
- □ $CAB + CB + C1$ *Commutative Law, twice*
- □ $C(AB + B + 1)$ *Distributive Law*
- □ $C(1 + AB + B)$ *Commutative Law, twice*
- □ $C1$ *Null Law*
- □ $1C$ *Commutative Law*
- □ C *Identity Law*

$\Rightarrow$ Often more than one way to do proof!