Homework

- ☐ Reading: Chapter 8
- ☐ Exercises: Chapter 8, all
- ☐ Due Friday, 9/28

COS 140: Foundations of Computer Science

Adders

Fall 2018

Adders

What is an adder?

- An adder is a logic circuit that adds binary numbers
- Could add two 1-digit numbers or two n-bit numbers

Why Study Adders?

- Interesting example of a combinational circuit.
  - Circuit whose output relies solely on its inputs.
  - Perform an important function for the computer.
    - Addition is also basis for other arithmetic functions in the computer (subtraction, multiplication, etc.)
    - Would like the function done in hardware so it is done quickly.

How Do We Do Addition?

1. Write down numbers that will be added using symbols from 0 to 9.
2. Use arithmetic facts to add numbers in a column. If more than 9, carry the most significant digit to the next column.

Numbers and Digital Logic

- Symbols will correspond to the 0 or 1 that is the input or output of the circuit. So, have 2 symbols to work with, not 10.
- Create a binary system that is like our digital system.

A Closer Look at Our Digital System

- Have 10 digits: 0–9
- Have "places" for 1's, 10's, 100's, 1000's, 10,000's, etc. that correspond to powers of 10.
  - $10^0 = 1; 10^1 = 10; 10^2 = 100; 10^3 = 1000; 10^4 = 10,000$
- To find the value of a number, add all the digits times their place values.
  - $359 = 9 \times 1 + 5 \times 10 + 3 \times 100$

In a Binary System...

- Have 2 digits: 0 and 1
- Places correspond to powers of 2:

2^0	1	2^4	16	2^8	256
2^1	2	2^5	32	2^9	512
2^2	4	2^6	64	2^{10}	1024
2^3	8	2^7	128	2^{11}	2048

- To find the value, add all the 1's and 0's times their place values.
  - Example from lecture: $10110 =$?

From decimal to binary

Given: n, a decimal number

1. First find the largest power of two less than n; let i be the exponent
2. Write down a 1, and $n = n$ minus that power of two
3. Decrement i to work on next-lower binary digit; if $i = 0$, we're done
4. If $2^i > n$, then there should be a 0 for that power of two; write that down, and go to 3
5. Else, if $2^i = n$, then write 0s for all the rest of the digits, and you're done
6. Otherwise ($2^i < n$), write a 1, since this power of 2 "fits" in n; $n = n - 2^i$, and go to 3

The algorithm

```
Algorithm Convert(d)
    Input: d, a decimal number
    Output: the binary version of d
    Let n be largest whole number such that 2^n ≤ d
    while n ≥ 0 do
        if d = 2^n then
            Output 1 followed by n − 1 0s
            return
        else if d < 2^n then
            Output 0
            n = n − 1
        else
            Output 1
            d = d − 2^n
            n = n − 1
        end if
    end while
End.
```


Example

	359	$2^8 < d < 2^9,\ \therefore n = 8 \Rightarrow$	1
-	256	Subtract $2^8 = 256,\ n = 7$	
	103	$2^7 > d;\ n = 6 \Rightarrow$	0
		$2^6 < d \Rightarrow$	1
-	64	Subtract $2^6 = 64,\ n = 5$	
	39	$2^5 < d \Rightarrow$	1
-	32	Subtract $2^5 = 32,\ n = 4$	
	7	$2^4 > d;\ n = 3 \Rightarrow$	0
		$2^3 > d;\ n = 2 \Rightarrow$	0
		$2^2 < d \Rightarrow$	1
-	4	Subtract $2^2 = 4,\ n = 1$	
	3	$2^1 < d \Rightarrow$	1
-	2	Subtract $2^1 = 2,\ n = 0$	
	1	$2^0 = d,\ \Rightarrow$	1

So $359_{10} = 101100111_2$

There are only 10 kinds of people in this world. Those that understand binary and those that don't.
– graduate student's T-shirt

Adders

Use Arithmetic Facts to Add Numbers

- Addition results from applying facts about arithmetic to numbers
- For the computer to use arithmetic facts, we need to construct a circuit.
- So: start with a truth table.
- Construct a truth table for all of the inputs, including the possible carry.

Truth Table for Addition

Carry-in	A	B	Carry-out	Sum
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Circuit from truth table

- Can we find a circuit for this? A minimal circuit?
- Karnaugh map for carry out:

C_in	A	B	C_out	Sum
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

AB

C_out	00	01	11	10
0			**1**	
1		**1**	**1**	**1**

AB + BC + AC

Karnaugh map for sum out:

C_in	A	B	C_out	Sum
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

AB

Sum	00	01	11	10
0		**1**		**1**
1	**1**		**1**	

~A~BC + ~AB~C +
ABC + A~BC

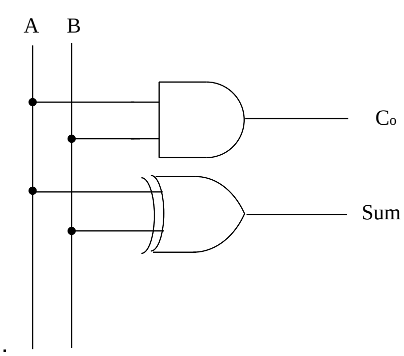
A better idea

- So minimization using Karnaugh maps, algebraic substitution – not so good!
- Can we do better?
- Maybe – inspect the truth table
- Things are simplified when we look at just A and B as inputs:

A	B	Carry	Sum
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

- Sum and carry – both correspond to basic operations/gates
- Sum $= A \oplus B$
- Carry $= AB$

Half-Adder



We can create a very simple circuit to add A and B.
Half-adder because only does half the job.

We need a *full adder* that adds $A + B + C_{in}$

Full Adder

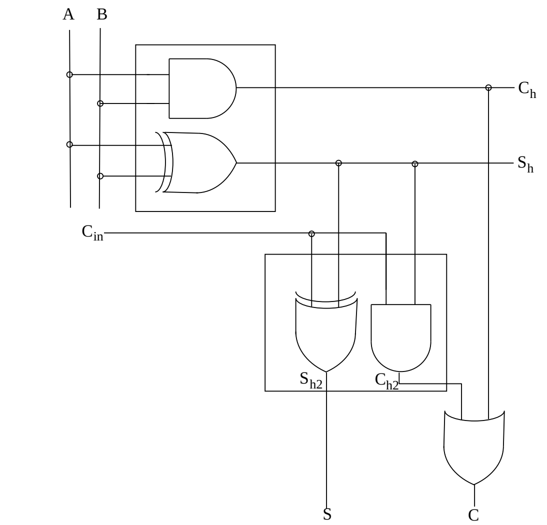
- A, B, $C_{in} \longrightarrow S$ (sum), C (carry out)
- Can we use a half-adder + additional logic get outputs?
- Half adder: A, $B \longrightarrow S_h$, C_h
- Generating S (sum):
  - $S = A + B + C_{in} = (A + B) + C_{in} = S_h + C_{in}$
  - Use another half-adder: S_h, $C_{in} \longrightarrow S_{h2} = S$

Full Adder

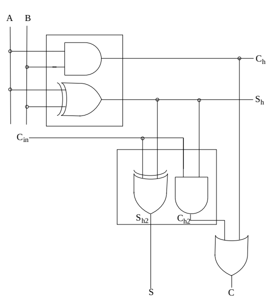
- When is C (carry out) $= 1$? When $A + B + C_{in} \geq 10_2$
  - Case 1: $A + B = 10_2$
    - Doesn't matter what C_{in} is: $C = 1$
    - In this case: $C_h = 1$
  - Case 2: $A + B = 1$ and $C_{in} = 1$
    - This means that $S_h = 1$, $C_{in} = 1$
    - In this case, carry out of second half-adder $C_{h2} = 1$
  - So $C = 1$ when either either or both half-adder carries is 1
  - $\therefore C = C_h \vee C_{h2}$

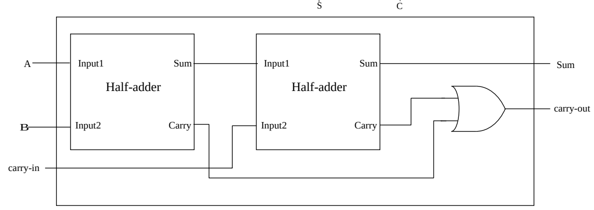
Full Adder

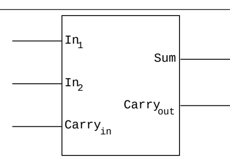
We can create a full adder by putting two half adders together as described above.



Full Adder







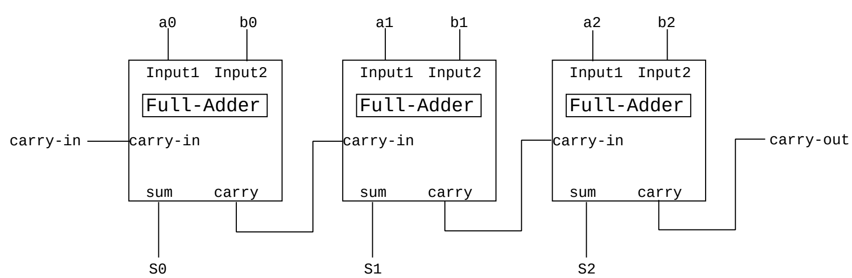
Multi-bit Adders

Creating Multi-Bit Adders

- Add multi-digit binary numbers using a full-adder for each bit.
- Problem: How to compute the carry-in for adder n?

Computing Carries: Ripple Carry

- Hook up required number of full adders.
- As carry is calculated, passed to next bit.



Computing Carries: Carry Lookahead

- ☐ First calculate what carry bits would be, based on previous bits
- ☐ Another way to specify carry-out is: carry-out of any full-adder (C_n) is true if:
  - carry-out of the first half-adder ($A_n B_n$) is true, or
  - either one of the inputs and the carry-in is true: $(A_n + B_n)C_{n-1}$
- ☐ So: $C_n = A_n B_n + (A_n + B_n)C_{n-1}$ – a *recurrence relation*

Computing Carries: Carry Lookahead

- ☐ We can calculate any carry using the recurrence relation:

$$C_n = A_n B_n + (A_n + B_n)C_{n-1}$$

  - $C_0 = A_0 B_0$, assuming no carry-in to low-order bit
  - $C_1 = A_1 B_1 + (A_1 + B_1)C_0 \Rightarrow$
    $C_1 = A_1 B_1 + (A_1 + B_1)A_0 B_0$
  - $C_2 = A_2 B_2 + (A_2 + B_2)C_1 \Rightarrow$
    $C_2 = A_2 B_2 + (A_2 + B_2)(A_1 B_1 + (A_1 + B_1)A_0 B_0)$
  - ...

Computing Carries: Carry Lookahead

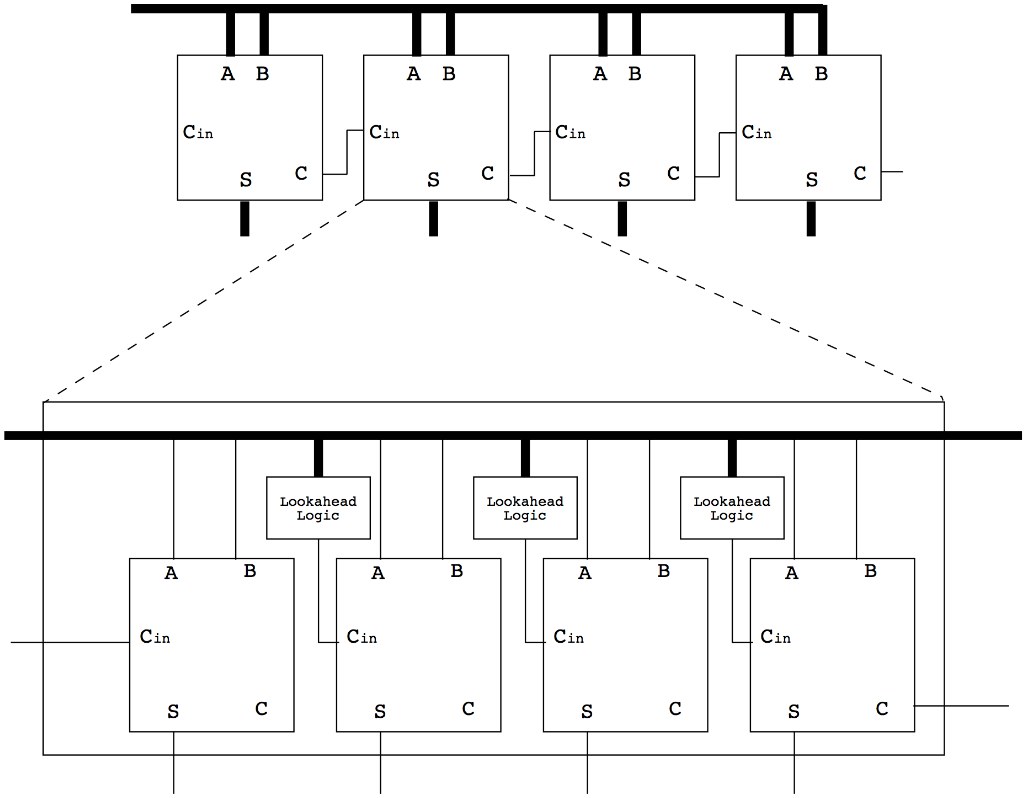
- Alternative could be done using $\oplus$:
  - For carry-out of the first full-adder (C_0), it's the carry-out of the full-adder's first half-adder OR'd with the carry-out of the second:
    $C_0 = C_{0_h} \vee C_{0_f} \Rightarrow$
    $C_0 = A_0B_0 \vee S_0C_{in} \Rightarrow$
    $C_0 = A_0B_0 \vee (A_0 \oplus B_0)C_{in}$
  - C_1 is computable based on C_0 in the same way: $C_1 = A_1B_1 \vee (A_1 \oplus B_1)C_0 \Rightarrow$
    $C_1 = A_1B_1 \vee (A_1 \oplus B_1)(A_0B_0 \vee (A_0 \oplus B_0)C_{in})$
  - Can generalize to n bits
- But better to keep with ANDs and ORs

Trade-offs for Types of Carry Propagation

- Ripple carry has larger delay.
  - Ripple carry has delay as more significant binary digits wait for results from less significant digits.
  - Carry lookahead uses a sum of products to get result for each carry, so only a two gate delay (have an AND layer and an OR layer).
- Complexity of circuit.
  - Ripple carry requires only connecting carry-out to next carry-in.
  - Number of AND gates and number of inputs to OR gate is on the order of the number of digits for carry lookahead (i.e., $O(n)$, where n is the number of digits)

Combining Ripple Carry and Carry Lookahead

- Minimize complexity of carry lookahead by only using it on a small number of bits in a group.
- Put groups together with a ripple carry.



What have we learned?

- Combinational circuits are important for computers
- Sometimes direct minimization of circuits via SOP may not be best...
- ...need to think outside the mechanistic box!
- Easy way of combining circuits may not be best way
  - Sometimes best way requires a lot of work to find
  - Sometimes "best" may not have a single meaning...
  - ...may have to trade off (e.g.) time for circuit complexity
- Sometimes what looks hard to implement (carry lookahead) may not be (2 layers of gates)